Chapter 3

Complex Numbers

H §2

The *complex representation* offers many mathematical advantages over trigonometric expression for oscillators.

Complex numbers arise from *imaginary numbers*. Since there is no real number solution for $\sqrt{-1}$, the imaginary number $\boldsymbol{i}$ is arbitrarily assigned as the solution, *i.e.,*

$$i = \sqrt{-1} \Longrightarrow i^2 = -1$$

Complex Number: A complex number z is an ordered pair of real numbers $[a, b] \equiv a + ib$:

a is the *real part* of z ($\mathrm{Re}\{z\}$) and b is the *imaginary part* ($\mathrm{Im}\{z\}$).

Complex Conjugate: The *complex conjugate* of a complex number $z = a + ib$ is defined as $z^* \equiv a - ib$, *i.e.,* simply replace i with $-i$.wherever it appears!

Complex Arithmetic: Given two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the following arithmetic rules apply:

1. **Equality:** $z_1 = z_2$ if and only if $a_1 = a_2$ *and* $b_1 = b_2$;

2. **Addition:** $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$, (add real and imaginary parts separately, $\mathrm{Re}\{z_1 + z_2\} = a_1 + a_2$, $\mathrm{Im}\{z_1 + z_2\} = b_1 + b_2$);

3. **Multiplication:**

$$\begin{aligned}
z_1 \cdot z_2 &= (a_1 + ib_1) \cdot (a_2 + ib_2) \\
&= a_1 a_2 + a_1(ib_2) + a_2(ib_1) + (ib_1)(ib_2) \\
&= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \\
\mathrm{Re}\{z_1 z_2\} &= a_1 a_2 - b_1 b_2 \\
\mathrm{Im}\{z_1 z_2\} &= a_1 b_2 + a_2 b_1;
\end{aligned}$$

4. **Reciprocal:** *(use this trick)* multiply z_2 by 1 in this form:

$$\frac{z_2^*}{z_2^*} = \frac{a_2 - ib_2}{a_2 - ib_2} = 1, \textit{ assuming that } z_2^* \neq 0, \textit{ and thus that } z_2 \neq 0$$

to obtain the reciprocal of z_2:

$$\begin{aligned}
\frac{1}{z_2}\cdot\frac{z_2^*}{z_2^*} &= \frac{1}{a_2+ib_2}\cdot\frac{a_2-ib_2}{a_2-ib_2} \\
&= \frac{a_2-ib_2}{a_2^2+b_2^2} \\
& \left(\frac{a_2}{a_2^2+b_2^2}\right)+i\left(\frac{-b_2}{a_2^2+b_2^2}\right) \\
&\Longrightarrow \operatorname{Re}\left\{\frac{1}{z_2}\right\} = \frac{a_2}{a_2^2+b_2^2} \\
&\Longrightarrow \operatorname{Im}\left\{\frac{1}{z_2}\right\} = -\frac{b_2}{a_2^2+b_2^2}
\end{aligned}$$

The magnitude and phase of the reciprocal are:

$$\begin{aligned}
\left|\frac{1}{z_2}\right| &= \sqrt{\left(\frac{a_2}{a_2^2+b_2^2}\right)^2+\left(-\frac{b_2}{a_2^2+b_2^2}\right)^2} \\
&= \sqrt{\frac{a_2^2+b_2^2}{(a_2^2+b_2^2)^2}} = \sqrt{\frac{1}{a_2^2+b_2^2}}
\end{aligned}$$

$$\Phi\left\{\frac{1}{z_2}\right\} = \tan^{-1}\left[\frac{\left(-\frac{b_2}{a_2^2+b_2^2}\right)}{\left(\frac{a_2}{a_2^2+b_2^2}\right)}\right] = \tan^{-1}\left[-\frac{b_2}{a_2}\right] = -\tan^{-1}\left[\frac{b_2}{a_2}\right] = -\Phi\{z_2\}$$

5. **Division:** Apply multiplication and the reciprocal to obtain:

$$\frac{z_1}{z_2} = \frac{a_1+ib_1}{a_2+ib_2}$$

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1}{z_2}\cdot\frac{z_2^*}{z_2^*} = \frac{a_1+ib_1}{a_2+ib_2}\cdot\frac{a_2-ib_2}{a_2-ib_2} = \frac{(a_1a_2+b_1b_2)+i(a_2b_1-a_1b_2)}{a_2^2+b_2^2} \\
\operatorname{Re}\left\{\frac{z_1}{z_2}\right\} &= \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} \\
\operatorname{Im}\left\{\frac{z_1}{z_2}\right\} &= \frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}
\end{aligned}$$

6. The real and imaginary parts of z can be expressed in terms of z and z^*

$$\begin{aligned}
\operatorname{Re}\{z\} &= \frac{1}{2}(z+z^*) \\
\operatorname{Im}\{z\} &= \frac{1}{2}(z-z^*)
\end{aligned}$$

The magnitude of z is defined as

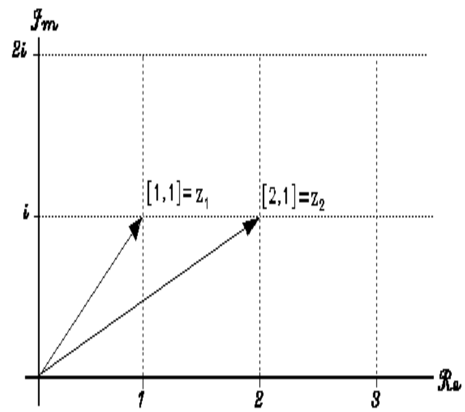
$$|z| \equiv \sqrt{z\cdot z^*}$$

3.1 Graphical Representation of Complex Numbers

As you learned in high-school algebra, any ordered pair of numbers can be located on a two-dimensional (2-D) graph, *e.g.,* using the Cartesian coordinates $[x, y]$. The y-axis becomes the *imaginary* axis, *i.e.,*all values along y are multiplied by $i = \sqrt{-1}$. Such a plot is sometimes called an *Argand* diagram.

For example,

$$
\begin{aligned}
z_1 &= 1+i \\
z_2 &= 2+i \\
&\Longrightarrow z_3 = z_1 + z_2 = 3+2i \\
&\Longrightarrow z_4 = z_1 - z_2 = -1
\end{aligned}
$$



Just as in algebra, we can also represent the Cartesian ordered pair $[a,b]$ in a polar notation $z_1 = (A_1, \phi_1)$, where A is the *magnitude* of the vector $[a,b]$ and ϕ is its polar angle (or *phase angle*):
We define the magnitude and phase of the complex number by:

$$
\begin{aligned}
\text{magnitude:} \qquad A_1 = |z_1| &\equiv \sqrt{z_1 \cdot z_1 *} \\
&= \sqrt{(a_1 + ib_1)(a_1 - ib_1)} \\
&= \sqrt{a_1^2 + b_1^2}
\end{aligned}
$$

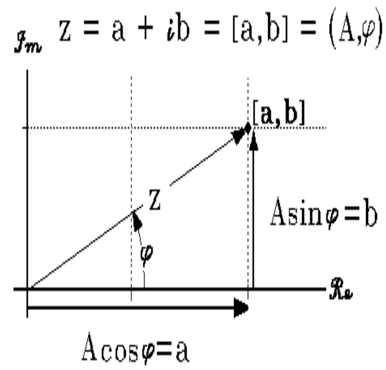
$$
\begin{aligned}
\text{phase:} \qquad \phi_1 &= \tan^{-1}\left[\frac{b_1}{a_1}\right] \\
&= \tan^{-1}\left[\frac{\mathrm{Im}\{z_1\}}{\mathrm{Re}\{z_1\}}\right]
\end{aligned}
$$

(*n.b.,* there is a subtle problem with this definition for the phase – the inverse tangent is defined on the interval $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, *i.e.,* the range is only π radians, whereas ϕ is defined over a range of 2π radians.)

$$
\begin{aligned}
\mathrm{Re}\{z\} &= \mathrm{Re}\{a+ib\} = a = A_1 \cos[\phi] \\
\mathrm{Im}\{z\} &= \mathrm{Im}\{a+ib\} = b = A_1 \sin[\phi]
\end{aligned}
$$

$$
\text{Magnitude (a real number)} \equiv |z| = \sqrt{a^2+b^2} = \sqrt{A_1^2 \cos^2\phi + A_1^2 \sin^2\phi} = \sqrt{A_1^2} = A_1
$$

$$
\begin{aligned}
z &= \mathrm{Re}\{z\} + i\,\mathrm{Im}\{z\} \\
&= A_1 \cos[\phi] + A_1(i \sin[\phi]) \\
&= A_1(\cos[\phi] + i \sin[\phi])
\end{aligned}
$$



3.2 Euler Relation – Complex Exponentials

H §2 pp.19-21, Schaum's Outline *Complex Variables* §1, Schaum's Outline *Optics* §1

Complex numbers are very conveniently denoted as exponentials; makes multiplication easy. Represent z in its polar form:

$$\begin{aligned} z &= (r, \phi) \\ &= r\left(\cos[\phi] + i\sin[\phi]\right) \\ &\equiv re^{i\phi} \end{aligned}$$

This expression arises from the *Euler relation*:

$$\boxed{\cos[\theta] + i\sin[\theta] = e^{i\theta}}$$

Proof: Consider $z = [r\cos\theta, r\sin\theta] = (r, \theta)$

$$\begin{aligned} z &\equiv r\left(\cos\theta + i\sin\theta\right) \\ dz &= \left(\cos\theta + i\sin\theta\right)dr + r\left(-\sin\theta d\theta + i\cos\theta d\theta\right) \\ &= \left(\cos\theta + i\sin\theta\right)dr + r\left(-\sin\theta + i\cos\theta\right)d\theta \\ &= r\left(\cos\theta + i\sin\theta\right)\frac{dr}{r} + r\left(i^2\sin\theta + i\cos\theta\right)d\theta \\ &= z\frac{dr}{r} + ir\left(\cos\theta + i\sin\theta\right)d\theta = z\left(\frac{dr}{r} + id\theta\right) \\ \frac{dr}{r} + id\theta &= \frac{dz}{z} \Longrightarrow \frac{dz}{z} = \int_0^z \frac{dz}{z} \equiv \log_e z = \int_0^r \frac{dr}{r} + i\int_0^\theta d\theta = \log_e r + i\theta \\ \log_e z &= \log_e r + i\theta \Longrightarrow e^{\log_e z} = e^{\log_e r + i\theta} = e^{\log_e r} + e^{i\theta}) \\ &\Longrightarrow z = [r\cos\theta, r\sin\theta] \Longrightarrow z = re^{i\theta} \end{aligned}$$

A different proof of Euler's relation, for those who know power-series expansions:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad [\textit{where } 0! \equiv 1] \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \Longrightarrow \lim_{\theta\to 0}\{\cos\theta\} = 1 \\
\sin\theta &= \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \Longrightarrow \lim_{\theta\to 0}\{\sin\theta\} = 0
\end{aligned}$$

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\
&= 1 + i\theta + \frac{i^2\theta^2}{2!} + i^2\frac{i\theta^3}{3!} + \frac{i^2 \cdot i^2 \cdot \theta^4}{4!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{i\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

As an aside, the approximations for cosine, sine, and tangent of small angles may be evaluated from the series:

$$\begin{aligned}
\lim_{\theta\to 0}\{\cos[\theta]\} &= 1 \\
\lim_{\theta\to 0}\{\sin[\theta]\} &= \theta \\
\lim_{\theta\to 0}\{\tan[\theta]\} &= \lim_{\theta\to 0}\left\{\frac{\sin[\theta]}{\cos[\theta]}\right\} = \theta
\end{aligned}$$

Graphs fo these three functions are compared in the figure:

Plots of $\theta, \sin[\theta]$, and $\tan[\theta]$ for $|\theta| \leq 1$ radian, showing that the three functions are approximately equal for $|\theta| \lesssim \frac{\pi}{10} \cong 0.31$ radians.

3.3 Arithmetic of Complex Exponentials

1. **equality:**

$$z_1 = A_1 e^{i\phi_1} \textit{ is equal to } z_2 = A_2 e^{i\phi_2} \textit{ if and only if } A_1 = A_2 \textit{and } \phi_1 = \phi_2$$

2. **addition:**

$$z_1 + z_2 = A_1 e^{i\phi_1} + A_2 e^{i\phi_2}$$

3. **multiplication:**

$$z_1 z_2 = A_1 e^{i\phi_1} A_2 e^{i\phi_2} = A_1 A_2 e^i (\phi_1 + \phi_2)$$

4. **division:**

$$\frac{z_1}{z_2} = \frac{A_1}{A_2} e^{i\phi_1} e^{-i\phi_2} = \frac{A_1}{A_2} e^{i(\phi_1 - \phi_2)}$$

3.3.1 De Moivre's Theorem

Generalization of multiplication of complex exponentials:

$$\begin{aligned} z^n &= [r(\cos\phi + i\sin\phi)]^n = r^n [\cos\phi + i\sin\phi]^n \\ &= r^n (\cos[n\phi] + i\sin[n\phi]) \quad \text{Proof by induction} \end{aligned}$$

A representation of $e^{-i\theta}$ can be derived using De Moivre's theorem or the series expansion of $e^{i\theta}$:

$$\begin{aligned} e^{-i\theta} &= e^{i(-\theta)} = 1 + i(-\theta) + \frac{i^2(-\theta)^2}{2!} + \frac{i^3(-\theta)^3}{3!} + \cdots \\ &= 1 - i\theta - \frac{\theta^2}{2!} + i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(-\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots\right) \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta - i\sin\theta \\ & \boxed{e^{-i\theta} = \cos\theta - i\sin\theta = \left(e^{+i\theta}\right)^*} \end{aligned}$$

Examples:

$$\begin{aligned} \phi &= 0 \Longrightarrow e^0 = 1 \quad \textit{because } \cos[0] = 1 \textit{ and } \sin[0] = 0 \\ \phi &= \frac{\pi}{2} \Longrightarrow e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\ \phi &= \pi \Longrightarrow e^{i\pi} = \cos\pi + i\sin\pi = -1 \end{aligned}$$

$$y[t] = A\cos[\omega t + \phi] = \mathrm{Re}\left\{Ae^{i(\omega t + \phi))}\right\} = \mathrm{Re}\{z[t]\}$$

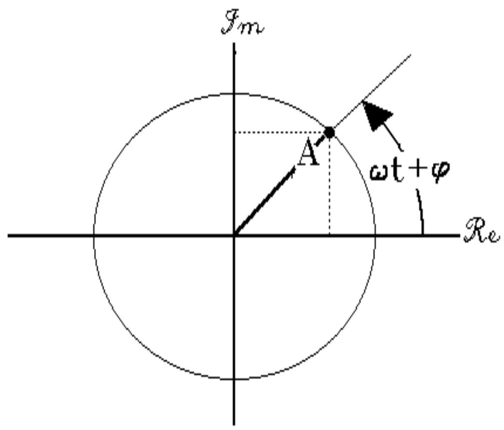
As will be discussed, products and sums of same-frequency harmonic oscillations are easily computed.

3.4 Description of Harmonic Oscillations via the Euler relation

To illustrate the utility of complex exponentials for describing harmonic oscillators, consider the action of $z[t] = Ae^{i\omega t}$ as a function of time:

$$\begin{aligned}
z[t=0] &= Ae^{i\cdot 0} = A \\
z\left[t = \frac{\pi}{4\omega} = \frac{T}{8}\right] &= Ae^{\frac{i\pi}{4}} = A\left(\cos\left[\frac{\pi}{4}\right] + i\sin\left[\frac{\pi}{4}\right]\right) = A\left(\frac{1}{\sqrt{2}}\right)(1+i) \\
z\left[t = \frac{\pi}{2\omega} = \frac{T}{4}\right] &= Ae^{\frac{i\pi}{2}} = A\left(\cos\left[\frac{\pi}{2}\right] + i\sin\left[\frac{\pi}{2}\right]\right) = A \cdot i \\
z\left[t = \frac{\pi}{\omega} = \frac{T}{2}\right] &= Ae^{i\pi} = A\left(\cos[\pi] + i\sin[\pi]\right) = -A \\
z\left[t = \frac{3\pi}{2\omega} = \frac{3T}{4}\right] &= Ae^{\frac{i3\pi}{2}} = A\left(\cos\left[\frac{3\pi}{2}\right] + i\sin\left[\frac{3\pi}{2}\right]\right) = A \cdot -1 = -A
\end{aligned}$$

As t increases, the complex function describes a circle of radius A about the origin.



If the vector rotates in the direction of $+\phi$ with increasing time, then the oscillation frequency is *positive*; if the vector rotates in direction of $-\phi$ with increasing time, the frequency is *negative.* The temporal frequency is proportional to the *rate of change* of phase. In words, the faster the oscillation, the more rapidly the phase changes:

$$\omega = \frac{\partial \Phi[t]}{\partial t}$$

where $\Phi[t]$ is the phase of the complex function. Since the phase has "units" of radians, its temporal derivative has dimensions of *radians per unit time.* The quantity ω is the *angular temporal frequency.* Since there are 2π radians per cycle, the angular temporal frequency may be converted to temporal frequency ν via:

$$\frac{\partial \Phi[t]}{\partial t}\left[\frac{\text{radians}}{\text{second}}\right] \times \frac{1}{2\pi\left[\frac{\text{radians}}{\text{cycle}}\right]} = \frac{1}{2\pi}\frac{\partial \Phi[t]}{\partial t}\left[\frac{\text{cycles}}{\text{second}}\right] \equiv \nu_0 \quad [Hz]$$

The temporal is proportional to the time derivative of the phase, which shows directly that the temporal frequency ν_0 is negative if the phase <u>decreases</u> with <u>increasing</u> time.

3.5 Oscillations as Projections of Circular Harmonic Motion

The sum of these two harmonic oscillations $\cos[\omega t]$ and $i\ \sin[\omega t]$ yields uniform circular motion. Because the sine term is imaginary, it is oriented at right angles to the (real) cosine term. The imaginary part of the motion can also be rewritten:

$$\begin{aligned}\sin[\omega t] &= \cos\left[\frac{\pi}{2}-\omega t\right]=\cos\left[-\left(\frac{\pi}{2}-\omega t\right)\right]=\cos\left[+\omega t-\frac{\pi}{2}\right]\ \text{(because cosine is even)}\\ &\Longrightarrow y[t]=\cos[\omega t]+i\ \cos\left[\omega t-\frac{\pi}{2}\right]\end{aligned}$$

Thus: uniform circular motion results from the addition of two harmonic oscillations at right angles and with a phase difference of $\frac{\pi}{2}$ radians $= 90°$.

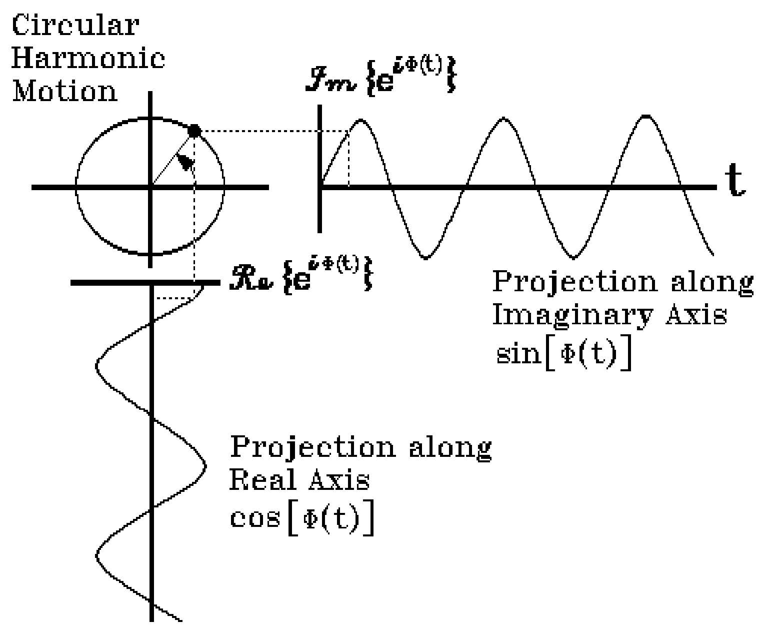
Conversely: the projection of uniform circular motion in any direction yields harmonic motion. The initial phase of the harmonic motion is determined by the azimuth of projection.

For example, when projecting onto the real axis, the information about variation along the imaginary axis is ignored:

$$Re\{y[t]\}=\cos[\omega t].$$

Projection onto the imaginary axis discards information about variation along the real axis, and the result:

$$Im\{y[t]\}=\sin[\omega t]=\cos\left[\omega t-\frac{\pi}{2}\right]$$



3.6 Phasor Notation for Oscillations

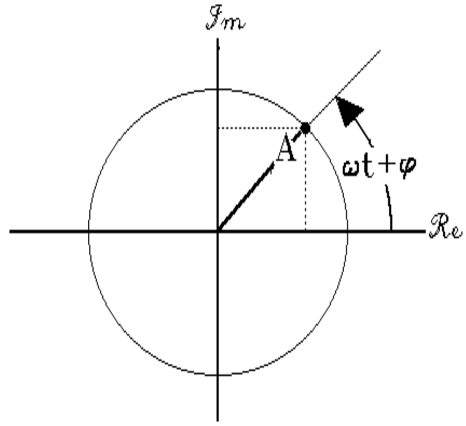
H §7.3

The interpretation of harmonic motion as a projection of uniform circular motion leads to a third method for representing oscillations – the phasor. Its use is quite popular in electrical engineering applications.

The phasor with magnitude A and phase $\Phi[t_0]$ is denoted by the polar vector $(A,\Phi[t_0])$ that describes the instantaneous position of the oscillator on the 2-D plot (Argand diagram). As time

progresses, the phasor of an oscillator rotates with period $T = \frac{1}{\nu} = \frac{2\pi}{\omega}$. Generally, the phasor picture portrays the amplitude and phase of the oscillator at a particular time t_0 (generally $t_0 = 0$ seconds).

Since the phasors of same-frequency oscillators rotate at the same rate, their relative phase is invariant. Therefore, the phasor picture is useful for describing the relative amplitudes and phases of two or more oscillators *with the same frequency*. Also, it is useful for finding the resultant of the superposition of the same-frequency oscillators, as will be shown.



3.7 Superposition of Oscillations

H§7,§14

When two (or more) oscillations (or waves) are present at the same location in a medium at the same time, the resultant motion is (obviously) some combination of the two component oscillations (or waves). The simplest combination of the components (and the most common for electromagnetic oscillations or waves) is the *superposition*, or sum. When the *principle of superposition* holds, the response is said to be *linear*, *i.e.,* the resultant $y[t]$ is the linear combination of the components $y_1[t] + y_2[t]$. The principle of superposition holds for acoustic and electromagnetic waves in most common situations (*e.g.,* EM waves in a vacuum).

3.7.1 Digression: Nonlinear Optics and Second-Harmonic Generation:

To help illustrate linear media and the principle of superposition, we will first consider an example where superposition is not valid. There are situations and media which can generate a resultant that is not a linear combination of the components. This effect has developed into the field of *nonlinear optics*. For example, a high-energy laser focused on one of a class of crystals (such as quartz or potassium dihydrogen phosphate – KDP) which generate some emerging energy proportional to *square* of the sum of the incident electric field E:

$$\begin{aligned} E[t] &\simeq (E_1 \cos[\omega_1 t] + E_2 \cos[\omega_2 t])^2 \\ &= E_1^2 \cos^2[\omega_1 t] + E_2^2 \cos^2[\omega_2 t] + 2E_1 E_2 \cos[\omega_1 t] \cos[\omega_2 t] \end{aligned}$$

As we will shortly demonstrate, the third term on the right-hand side can also be written:

$$2E_1 E_2 \cos[\omega_1 t] \cos[\omega_2 t] = E_1 E_2 \{\cos[(\omega_1 + \omega_2) t] + \cos[(\omega_1 - \omega_2) t]\}.$$

Thus, electromagnetic interactions in nonlinear media can generate components with sum and difference frequencies. If both input beams have the same frequency ω and wavelength λ, there will be an output component with frequency $\omega' = 2\omega, \lambda' = \frac{\lambda}{2}$. For example, a laser rod composed of Yttrium-Aluminum-Garnet doped with Neodymium (a *Nd:YAG* laser) can lase to make a beam

with $\lambda = 1.06\,\mu$m (in the "near-infrared" region of the spectrum). If the laser beam has sufficient energy and is directed onto a crystal that is has a strong "nonlinear" response, an output beam may be produced at the "doubled wavelength" $\lambda' = 0.53\,\mu$m, *i.e.,* visible green light. Such an effect is called *second-harmonic generation* and is a very active research area in quantum optics.

Though nonlinear effects are of great interest in optics today, we will just consider situations where the principle of superposition is valid – the output is the sum of the component terms.

3.8 Superposition of Same-Frequency Oscillations – Trigonometric Notation

Consider the linear superposition of two oscillations with the same frequency and different amplitudes and phases:

$$\begin{aligned} y_1[t] &= A_1 \cos[\omega t + \phi_1] \\ y_2[t] &= A_2 \cos[\omega t + \phi_2] \Longrightarrow y[t] = y_1[t] + y_2[t] \end{aligned}$$

The trigonometric solution of the resultant $y[t]$ can be found as follows:

$$\begin{aligned} y_1 + y_2 &= A_1(\cos[\omega t]\cos[\phi_1] - \sin[\omega t]\sin[\phi_2]) + A_2(\cos[\omega t]\cos[\phi_2] - \sin[\omega t]\sin[\phi_2]) \\ &= \cos[\omega t](A_1\cos[\phi_1] + A_2\cos[\phi_2]) - \sin[\omega t](A_1\sin[\phi_1] + A_2\sin[\phi_2]) \end{aligned}$$

Since real parts add to real parts, etc., we can define the real and imaginary parts of the resultant:

$$\begin{aligned} \mathrm{Re}\{(\mathcal{A},\phi)\} &= \mathcal{A}\cos\phi = A_1\cos[\phi_1] + A_2\cos[\phi_2] \\ \mathrm{Im}\{(\mathcal{A},\phi)\} &= \mathcal{A}\sin\phi = A_1\sin[\phi_1] + A_2\sin[\phi_2]\,. \end{aligned}$$

The squared magnitude of the result is:

$$\begin{aligned} (\mathcal{A}\sin\phi)^2 + (\mathcal{A}\cos\phi)^2 &= \mathcal{A}^2 = (A_1\sin[\phi_1] + A_2\sin[\phi_2])^2 + (A_1\cos[\phi_1] + A_2\cos[\phi_2])^2 \\ &\Longrightarrow \mathcal{A} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}, \end{aligned}$$

and phase:

$$\begin{aligned} \frac{\mathcal{A}\sin\phi}{\mathcal{A}\cos\phi} &= \tan\phi = \frac{A_1\sin[\phi_1] + A_2\sin[\phi_2]}{A_1\cos[\phi_1] + A_2\cos[\phi_2]} \\ &\Longrightarrow \phi = \tan^{-1}\left[\frac{A_1\sin[\phi_1] + A_2\sin[\phi_2]}{A_1\cos[\phi_1] + A_2\cos[\phi_2]}\right]. \end{aligned}$$

Consider some simple cases:

1. $A_1 = A_2, \phi_1 = \phi_2 \Longrightarrow$ same amplitude, same phase:

$$\begin{aligned} A_1\cos[\omega t + \phi_1] + A_1\cos[\omega t + \phi_1] &= 2A_1\cos[\omega t + \phi_1] \Longrightarrow A = 2A_1,\ \phi = \phi_1 \\ \mathcal{A}^2 &= A_1^2 + A_1^2 + 2A_1A_1\cos(\phi_1 - \phi_1) = 2A_1^2 + 2A_1^2\cos(0) = 4A_1^2 \\ & \boxed{\mathcal{A} = 2A_1} \end{aligned}$$

$$\begin{aligned} \tan\phi &= \left[\frac{2A_1\sin[\phi_1]}{2A_1\cos[\phi_1]}\right] \\ &\Longrightarrow \boxed{\phi = \phi_1} \end{aligned}$$

Addition of two identical oscillations gives a resultant with twice the amplitude and the same phase, as expected.

2. $A_1 = A_2, \phi_2 = (\phi_1 - \pi) \Longrightarrow$ same amplitude, phase difference of π radians:

$$\begin{aligned} \mathcal{A}^2 &= A_1^2 + A_1^2 + 2A_1A_1 \cos\left[(\phi_1 - (\phi_1 - \pi)\right] \\ &= 2A_1^2 + 2A_1^2 \cos[\pi] = 2A_1^2 - 2A_1^2 = 0 \\ \Longrightarrow & \mathcal{A} = 0 \end{aligned}$$

$$\phi = \tan^{-1}[\pi] = \pm\infty, \text{ but } \phi \text{ is irrelevant since amplitude } \mathcal{A} = 0$$

Addition of two oscillations with same amplitude but out of phase by $\pm\pi$ radians gives zero output, also as expected.

3. $A_1 = A_2, \phi_2 = \phi_1 + \frac{\pi}{2} \Longrightarrow$ same amplitude, phase difference of $+\frac{\pi}{2}$ radians.
The resultant has magnitude :

$$\begin{aligned} \mathcal{A}^2 &= A_1^2 + A_1^2 + 2A_1^2 \cos\left[\phi_1 - \phi_1 - \frac{\pi}{2}\right] \\ &= 2A_1^2\left(1 + \cos\left[\frac{\pi}{2}\right]\right) = 2A_1^2 \Longrightarrow \mathcal{A} = \sqrt{2}A_1, \end{aligned}$$

and phase:

$$\tan\phi = \left[\frac{A_1 \sin[\phi_1] + A_1 \sin(\phi_1 + \frac{\pi}{2})}{A_1 \cos[\phi_1] + A_1 \cos(\phi_1 + \frac{\pi}{2})}\right]$$

Since: $\cos\left[\phi_1 + \frac{\pi}{2}\right] = -\sin[\phi_1]$, $\sin\left[\phi_1 + \frac{\pi}{2}\right] = \cos[\phi_1]$,

$$\begin{aligned} \Longrightarrow \quad \tan\phi = \left[\frac{\cos[\phi_1] + \sin[\phi_1]}{\cos[\phi_1] - \sin[\phi_1]}\right] &= \left[\frac{\cos[\phi_1](1 + \tan[\phi_1])}{\cos[\phi_1](1 - \tan[\phi_1])}\right] \\ = \left[\frac{\tan\left[\frac{\pi}{4}\right] + \tan\phi_1}{1 - \tan\left[\frac{\pi}{4}\right]\tan\phi_1}\right], & \text{ since } \tan\left[+\frac{\pi}{4}\right] = 1. \end{aligned}$$

Now we cheat, from a table of trigonometric properties, we can find that:

$$\tan[\alpha + \beta] = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Thus:

$$\left[\frac{\tan\left[\frac{\pi}{4}\right] + \tan[\phi_1]}{1 - \tan\left[\frac{\pi}{4}\right]\tan[\phi_1]}\right] = \tan\left[\phi_1 + \frac{\pi}{4}\right].$$

and the phase of the resultant is:

$$\phi = \phi_1 + \frac{\pi}{4} \text{ (if } \phi_1 = 0, \text{ then } \phi = +\frac{\pi}{4})$$

If you add two oscillations with the same amplitude and a phase difference of +, the resultant has the Pythagorean amplitude $\mathcal{A} = \sqrt{A_1^2 + A_1^2}$ and a phase angle midway between those of the components.

3.9 Superposition of Same-Frequency Oscillations – Phasor Representation

Phasors are useful for computing the magnitude and phase resulting of the superposition (sum) of two (or more) oscillators with the same frequency. The resultant of the superposition of two oscillators is the vector sum of the phasors defining the two oscillators:

$$\begin{aligned} y_1[t] &= A_1 \sin[\omega t + \phi_1] \equiv (A_1, \phi_1) \\ y_2[t] &= A_2 \sin[\omega t + \phi_2] \equiv (A_2, \phi_2). \end{aligned}$$

The resultant phasor is $(\mathcal{A},\phi) = (A_1,\phi_1) + (A_2,\phi_2)$. The magnitude can be computed by adding the real and imaginary parts separately:

$$\begin{aligned}
\mathrm{Re}\{\mathcal{A}\} &= \mathrm{Re}\{A_1\} + \mathrm{Re}\{A_2\} = A_1\cos[\phi_1] + A_2\cos[\phi_2] \\
\mathrm{Im}\{\mathcal{A}\} &= \mathrm{Im}\{A_1\} + \mathrm{Im}\{A_2\} = A_1\sin[\phi_1] + A_2\sin[\phi_2]
\end{aligned}$$

Since the two oscillators have the same frequency ω, the relative phase of the two oscillators is invariant, and thus the relative initial phase is sufficient to compute the relative phase of the resultant.

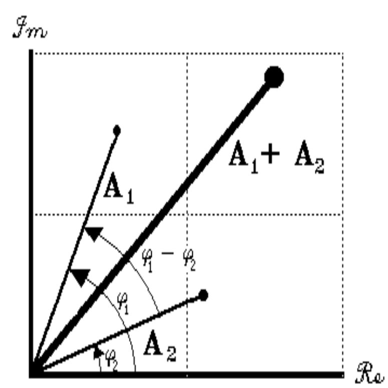
$$\phi = \tan^{-1}\left[\frac{A_1\sin[\phi_1] + A_2\sin[\phi_2]}{A_1\cos[\phi_1] + A_2\cos[\phi_2]}\right].$$

n.b., if the oscillators have different frequencies, the relative phase $\Phi_1[t] - \Phi_2[t]$ varies with time and the phasor picture is not useful.

The magnitude also may be computed by using the law of cosines:

$$\begin{aligned}
\mathcal{A}^2 &= A_1^2 + A_2^2 - 2A_1A_2\cos[\phi_1 - (\phi_2 - \pi)] \\
&= A_1^2 + A_2^2 - 2A_1A_2\cos[\pi - (\phi_2 - \phi_1)] \\
&= A_1^2 + A_2^2 + 2A_1A_2\cos[\phi_2 - \phi_1] \\
&= A_1^2 + A_2^2 + 2A_1A_2\cos[\phi_1 - \phi_2],
\end{aligned}$$

where the last step follows because $\cos[\theta] = \cos[-\theta]$.



3.10 Superposition of Same-Frequency Oscillations – Complex Notation

Consider the complex representation of two oscillators with the same frequency ω:

$$\begin{aligned}
y_1[t] &= A_1 e^{i[\omega t + \phi_1]} \\
y_2[t] &= A_2 e^{i[\omega t + \phi_2]}
\end{aligned}$$

$$\begin{aligned}
y[t] &\equiv \mathcal{A}e^{i(\omega t+\phi)} = e^{i\omega t}\mathcal{A}e^{i\phi} \\
&= y_1[t] + y_2[t] = A_1 e^{i[\omega t+\phi_1]} + A_2 e^{i[\omega t+\phi_2]} \\
&= e^{i\omega t}[A_1 e^{i\phi_1} + A_2 e^{i\phi_2}] = e^{i\omega t}\mathcal{A}e^{i\phi} \\
\Longrightarrow &\ \mathcal{A}e^{i\phi} = A_1 e^{i\phi_1} + A_2 e^{i\phi_2}
\end{aligned}$$

n.b., The resultant oscillation has the same frequency as the components frequency ω

The last line represents the sum of two phasors: $(A_1, \phi_1), (A_2, \phi_2)$. This was solved on the previous page:

$$\boxed{\mathcal{A} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}}$$

$$\boxed{\tan[\phi] = \left[\frac{A_1\sin[\phi_1] + A_2\sin[\phi_2]}{A_1\cos[\phi_1] + A_2\cos[\phi_2]}\right]}$$

THE SUPERPOSITION OF TWO SAME-FREQUENCY OSCILLATIONS IS AN OSCILLATION OF THAT FREQUENCY

3.11 Superposition of Many Same-Frequency Oscillators

Since the sum of two same-frequency oscillations is a harmonic oscillation of that frequency, clearly the sum of N same-frequency oscillations must also be a harmonic oscillation of that frequency. This is easy to prove using complex notation:

$$\begin{aligned}
y_n[t] &= A_n e^{i(\omega_0 t+\phi_n)} \\
y[t] &= \sum_{n=1}^{N} A_n e^{i(\omega_0 t+\phi_n)} = e^{i\omega_0 t}\sum_{n=1}^{N} A_n e^{i\phi_n} \equiv e^{i\omega_0 t}\left(\mathcal{A}e^{i\Phi}\right).
\end{aligned}$$

The resultant oscillation has amplitude $\mathcal{A}$ and phase Φ, and hence the phasor $(\mathcal{A}, \Phi)$.

$$\mathcal{A}e^{i\Phi} = \sum_{n=1}^{N} A_n e^{i\phi_n}$$

$$\begin{aligned}
\mathrm{Re}\{\mathcal{A}e^{i\Phi}\} &= \mathrm{Re}\left\{\sum_{n=1}^{N} A_n e^{i\phi_n}\right\} = \sum_{n=1}^{N} A_n\cos[\phi_n] \\
\mathrm{Im}\{\mathcal{A}e^{i\Phi}\} &= \mathrm{Im}\left\{\sum_{n=1}^{N} A_n e^{i\phi_n}\right\} = \sum_{n=1}^{N} A_n\sin[\phi_n]
\end{aligned}$$

By the Pythagorean theorem:

$$\mathcal{A}^2 = [Re\{\mathcal{A}\}]^2 + [Im\{\mathcal{A}\}]^2 = \left[\sum_{n=1}^{N} A_n\cos[\phi_n]\right]^2 + \left[\sum_{n=1}^{N} A_n\sin[\phi_n]\right]^2$$

Look at the real part:

$$\begin{aligned}
\left[\sum_{n=1}^{N} A_n \cos[\phi_n]\right]^2 &= \left[\sum_{j=1}^{N} A_j \cos[\phi_j]\right]\left[\sum_{k=1}^{N} A_k \cos[\phi_k]\right] \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N} A_j A_k \cos[\phi_j]\cos[\phi_k] \\
&= A_1^2\cos^2[\phi_1] + A_1A_2\cos[\phi_1]\cos[\phi_2] + A_2A_1\cos[\phi_2]\cos[\phi_1] + A_2^2\cos^2[\phi_2] + \cdots \\
&= A_1^2\cos^2[\phi_1] + A_2^2\cos^2[\phi_2] + \cdots + 2A_1A_2\cos[\phi_1]\cos[\phi_2] + 2A_2A_3\cos[\phi_2]\cos[\phi_3] + \cdots \\
&= \sum_{n=1}^{N} A_n^2\cos^2[\phi_n] + \sum_{j\neq k}^{N} A_jA_k\cos[\phi_j]\cos[\phi_k] \\
&= \sum_{n=1}^{N} A_n^2\cos^2[\phi_n] + 2\sum_{j>k}^{N} A_jA_k\cos[\phi_j]\cos[\phi_k],\ i.e.,\ j=[2,N],\ k=[1,N-1]
\end{aligned}$$

where the last two expressions are equivalent sums of the terms with $j = k$ and with $j \neq k$. The treatment for the imaginary part is identical:

$$\begin{aligned}
\left[\sum_{n=1}^{N} A_n \sin[\phi_n]\right]^2 &= \sum_{j=1}^{N} A_j^2\sin^2[\phi_j] + \sum_{j\neq k}^{N} A_jA_k\sin[\phi_j]\sin[\phi_k] \\
&= \sum_{j=1}^{N} A_j^2\sin^2[\phi_j] + 2\sum_{j>k}^{N} A_jA_k\sin[\phi_j]\sin[\phi_k]
\end{aligned}$$

Therefore the square of the resulting magnitude may be written as:

$$\begin{aligned}
\mathcal{A}^2 &= \left[\sum_{n=1}^{N} A_n\cos[\phi_n]\right]^2 + \left[\sum_{n=1}^{N} A_n\sin[\phi_n]\right]^2 + 2\left[\sum_{j>k}^{N} A_jA_k\cos[\phi_j]\cos[\phi_k] + \sum_{j>k}^{N} A_jA_k\sin[\phi_j]\sin[\phi_k]\right] \\
&= \sum_{n=1}^{N} A_{n^2}\left[\cos^2[\phi_n] + \sin^2[\phi_n]\right] + 2\sum_{j>k}^{N} A_jA_k\left(\sin[\phi_j]\ \ \sin[\phi_k] + \cos[\phi_j]\ \ \cos[\phi_k]\right)
\end{aligned}$$

Since the phase angles are randomly distributed, the phase angle of the resultant is randomly distributed as well – therefore, no prediction of the phase can be made.

$$\mathcal{A}^2 = \sum_{n=1}^{N} A_{n^2} + 2\sum_{j>k}^{N} A_jA_k\ \cos[\phi_j - \phi_k]$$

3.12 Superposition of Randomly Phased Oscillators

Special Case I: The oscillators have identical amplitudes ($A_j = A_k \equiv A_0$) and phases that are *randomly distributed* over the full domain of possible phase

Random phases $\Longrightarrow$ $[\phi_j]$ is randomly distributed in $[0, 2\pi)$ (i.e., $0 \leq \phi < 2\pi$) or $[-\pi, +\pi)$ ($-\pi \leq \phi < +\pi$).

$\Longrightarrow [\phi_j] - [\phi_k]$ is randomly distributed in $[-2\pi, 2\pi)$, and so is randomly distributed in $[0, 2\pi)$

$\Longrightarrow \cos[\phi_j - \phi_k]$ is randomly distributed in $[-1, 1]$

$$\Longrightarrow \mathcal{A}^2 = \sum_{n=1}^{N} A_0^2 + 2\cdot A_0^2\sum_{j>k}^{N}\cos[\phi_j - \phi_k]$$

Since $\cos[\phi_j - \phi_k]$ is randomly distributed over the interval $[-1, 1]$,

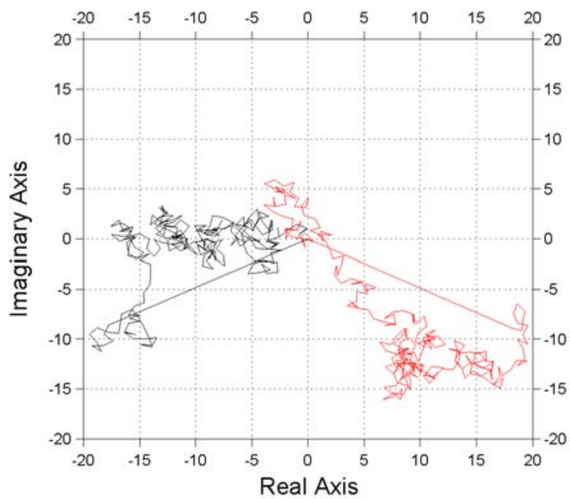
$$\sum_{j>k}^{N} \cos\left[\phi_j - \phi_k\right] \simeq 0$$

and

$$\begin{aligned} \mathcal{A}^2 &= \sum_{n=1}^{N} A_0^2 = N \cdot A_0^2 \\ &\Longrightarrow \boxed{\mathcal{A} = \left(\sqrt{N}\right) A_0} \end{aligned}$$

The phase of the sum of the random-phase oscillators cannot be predicted.

Recall that the *energy* of the oscillator is proportional to A^2, so if the phases are random, the total energy is the sum of the individual energies, as expected. Note that the total amplitude is $\sqrt{N}$ times as large as the individual amplitude. Randomly phased oscillators are said to be *incoherent*.



Two examples of superposition of randomly phased oscillators, showing resultant magnitudes.

3.13 Superposition of Nonrandomly Phased Oscillators

Special Case II: Amplitudes AND phases are equal, *i.e.*, $A_j = A_k = A_0$ and $\left[\phi_j\right] = [\phi_k] = \phi_0$

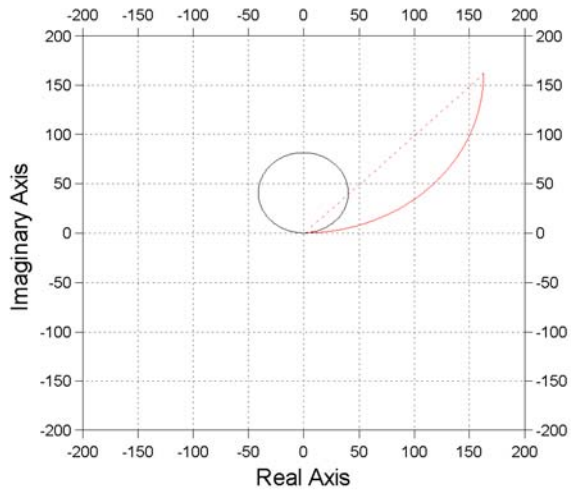
$$\begin{aligned} I &= \mathcal{A}^2 = N \cdot A_0^2 + 2 \cdot A_0^2 \sum_{j=2}^{N} \cos(\phi_0 - \phi_0) \\ &= N \cdot A_0^2 + 2 \cdot A_0^2 \sum_{j=2}^{N} 1 \\ &= A_0^2(N + 2(N-1)) \\ &= (3N-2)A_0^2 \end{aligned}$$

Examples:

$$
\begin{aligned}
N &= 1 \Longrightarrow I = A_0^2 \equiv I_0, \text{ one oscillator} \\
N &= 2 \Longrightarrow I = 4A_0^2 = 4I_0 \Longrightarrow 4 \times \text{energy of one oscillator} \\
N &= 3 \Longrightarrow I = 7A_0^2 = 7I_0 \\
N &= 4 \Longrightarrow I = 10A_0^2 = 10I_0
\end{aligned}
$$

n.b., $I > NI_0$, the intensity of the sum of N in-phase oscillators is larger than expected, *i.e.,* the noise is louder, or the light is brighter. Of course, energy must be conserved, so if the signal is "louder" or "brighter" at some locations, it must be "less loud" or "dimmer" at other locations.

If the phase relationship between the component oscillators is well-defined, the oscillators are *coherent*.



Two examples of the sum of nonrandomly phased oscillators. In one case (shown in black), the sum yields a "null output" (resulting magnitude is 0). In the other case, the sum is nonzero.

3.14 Superposition of Oscillations with Different Frequencies

3.14.1 Complex Notation

H§7.5, HR §20

We have just seen that the superposition of any number of same-frequency oscillators is an oscillation with that frequency. When superposing a number of oscillators with different frequencies, the situation is quite different – (almost) any periodic function can be synthesized from the summation of harmonic terms. This is the principle of Fourier analysis.

Simple Example – Addition of two oscillators of same amplitude A_0, same phase ϕ_0, different frequencies ω_1 and ω_2 :

$$
\begin{aligned}
y[t] &= y_1[t] + y_2[t] = A_0 \cos[\omega_1 t] + A_0 \cos[\omega_2 t] \\
&= A_0 (\cos[\omega_1 t] + \cos[\omega_2 t]) \\
&= \text{Re}\left\{A_0 \left[e^{i\omega_1 t} + e^{i\omega_2 t}\right]\right\}
\end{aligned}
$$

Note that:

$$\begin{aligned}
\Longrightarrow \quad & e^{+i\omega_1 t} + e^{+i\omega_2 t} = e^{+i\frac{\omega_1}{2}t} \cdot e^{+i\frac{\omega_1}{2}t} + e^{+i\frac{\omega_2}{2}t} \cdot e^{+i\frac{\omega_2}{2}t} \\
= \quad & \left(e^{+i\frac{\omega_1}{2}t} \cdot e^{+i\frac{\omega_1}{2}t}\right)\left(e^{+i\frac{\omega_2}{2}t} \cdot e^{-i\frac{\omega_2}{2}t}\right) + \left(e^{+i\frac{\omega_1}{2}t} \cdot e^{-i\frac{\omega_1}{2}t}\right)\left(e^{+i\frac{\omega_2}{2}t} \cdot e^{i\frac{\omega_2}{2}t}\right) \\
= \quad & \left(e^{+i\frac{\omega_1}{2}t} \cdot e^{+i\frac{\omega_2}{2}t}\right)\left(e^{+i\frac{\omega_1}{2}t} \cdot e^{-i\frac{\omega_2}{2}t}\right) + \left(e^{+i\frac{\omega_1}{2}t} \cdot e^{+i\frac{\omega_2}{2}t}\right)\left(e^{-i\frac{\omega_1}{2}t} \cdot e^{+i\frac{\omega_2}{2}t}\right) \\
= \quad & \left(e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right)\left(e^{+i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right) + \left(e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right)\left(e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right) \\
= \quad & \left(e^{+i\left(\frac{\omega_1-\omega_2}{2}\right)t} + e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)t}\right)\left(e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right) \\
= \quad & 2\ \cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\left(e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & y[t] = \mathrm{Re}\left\{A_0\left[e^{i\omega_1 t} + e^{i\omega_2 t}\right]\right\} \\
= \quad & A_0\,\mathrm{Re}\left\{e^{i\omega_1 t} + e^{i\omega_2 t}\right\} \\
= \quad & A_0\,\mathrm{Re}\left\{2\ \cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\left(e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right)\right\} \\
= \quad & 2A_0\ \cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\mathrm{Re}\left\{e^{+i\left(\frac{\omega_1+\omega_2}{2}\right)t}\right\} \\
= \quad & 2A_0\ \cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right] \cdot \cos\left[\left(\frac{\omega_1+\omega_2}{2}\right)t\right]
\end{aligned}$$

By defining an average and a modulation (angular) frequency:

$$\begin{aligned}
\Omega_{avg} &\equiv \frac{\omega_1+\omega_2}{2} \\
\Omega_{\mathrm{mod}} &\equiv \frac{\omega_1-\omega_2}{2},
\end{aligned}$$

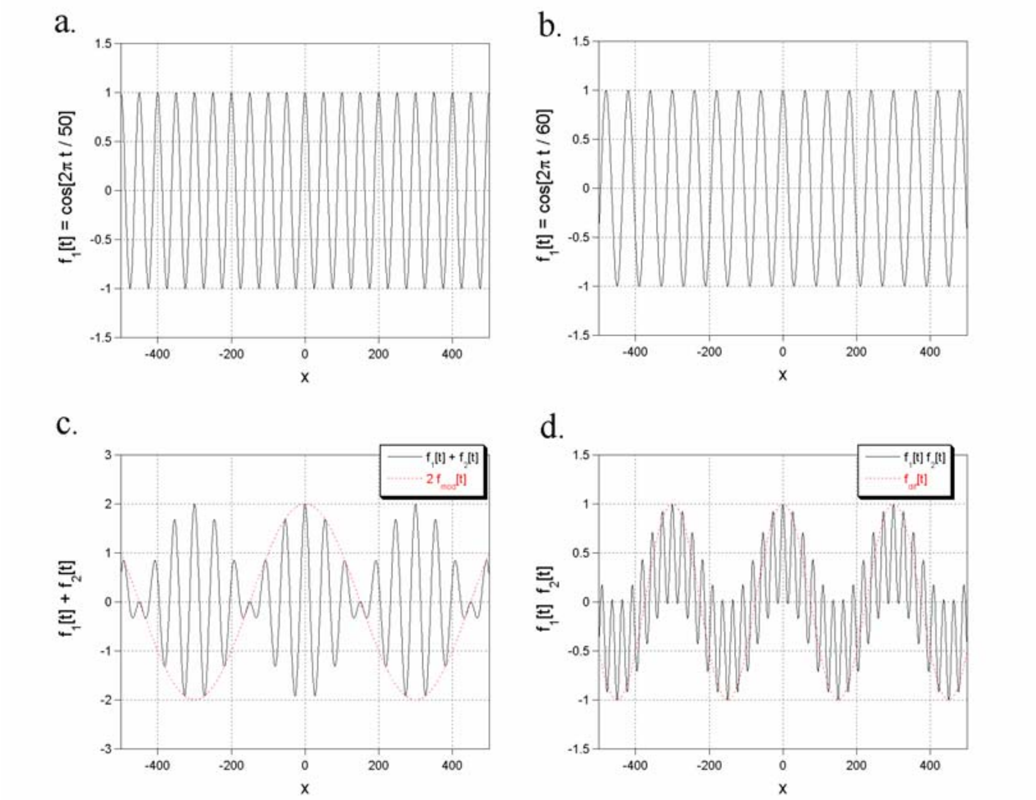
we obtain: $\boxed{y[t] = 2A_0\cos[\Omega_{avg}t]\ \ \cos[\Omega_{\mathrm{mod}}t]}$

In words, the sum of two harmonic oscillations with different frequencies ω_1 and ω_2 yields the product of two harmonic oscillations, one with the average frequency $\Omega_{avg} = \frac{\omega_1+\omega_2}{2}$, and one with the so-called modulation frequency $\Omega_{\mathrm{mod}} = \frac{\omega_1+\omega_2}{2}$.

Both the product and sum of different-frequency sinusoids yield results that are not harmonic. The former is equivalent to the sum of sinusoids at the sum and difference frequencies, while the sum is equivalent to the product of sinusoids at Ω_{avg} and Ω_{mod}. The periods of the superposition are T_{avg} and T_{mod}, where $T_{\mathrm{mod}} > T_{avg}$. The slower period T_{mod} is the source of the phenomenon known commonly as *beats*, from its musical context, though this kind of pattern is seen (heard?) in many other situations as well. Low-frequency Moiré fringes are seen when two periodic patterns are overlaid are examples. The phenomenon of *aliasing* in digital signal/image processing is closely related.

The converse is also true: the product of two periodic signals can be expressed as the sum of two other oscillations: the heterodyning operation in radio is an example. AM radio signals are broadcast at frequencies $560\,\mathrm{kHz} \le \nu_1 \le 1600\,\mathrm{kHz}$. To render the signals audible, they are beat down by multiplying by an *intermediate frequency* (IF) ν_2. Two signals result: one with frequency $\nu_1+\nu_2$ and one with $\nu_1-\nu_2$. Judicious choice of ν_2 puts the lower-frequency *sideband* in the audible range. The upper sideband is removed by a filter which passes only low frequencies (*low-pass filter*).

Illustrations: Consider the product and sum of two harmonic oscillations with angular frequencies $\nu_1 = \frac{1}{50}$ and $\nu_2 = \frac{1}{60}$ cycles per unit length, so the corresponding temporal periods are $T_1 = 50$ and $T_2 = 60$. These are illustrated below:



Sum and product of oscillations: (a) $f_1[t] = \cos\left[2\pi \frac{t}{50}\right]$, (b) $f_2[t] = \cos\left[2\pi \frac{t}{60}\right]$, (c) $f_1[t] + f_2[t]$, also showing modulation wave, (d) $f_1[t] \times f_2[t]$, showing different-frequency wave.

The sum of these two oscillations is:

$$\begin{aligned}
\cos[2\pi\nu_1 t] + \cos[2\pi\nu_2 t] &= 2\cos\left[2\pi\left(\frac{\nu_1+\nu_2}{2}\right)t\right]\cdot\cos\left[2\pi\left(\frac{\nu_1-\nu_2}{2}\right)t\right] \\
&= 2\cos\left[2\pi\left(\frac{\frac{1}{50}+\frac{1}{60}}{2}\right)t\right]\cdot\cos\left[2\pi\left(\frac{\frac{1}{14}-\frac{1}{18}}{2}\right)t\right] \\
&\simeq 2\cos\left[2\pi\frac{t}{54.545}\right]\cdot\cos\left[2\pi\frac{t}{600}\right] \\
\nu_{avg} &\simeq \frac{1}{54.545},\ \nu_{\text{mod}} = \frac{1}{600}
\end{aligned}$$

The "slowly" varying term with period 600 is generally more visible.

The product of ther two sinusoids may be written as the scaled sum of sinusoids at the *sum* and *difference frequencies*:

$$\begin{aligned}
\cos[2\pi\nu_1 t]\cdot\cos[2\pi\nu_2 t] &= \frac{1}{2}\cos[2\pi(\nu_1+\nu_2)t] + \frac{1}{2}\cos[2\pi(\nu_1-\nu_2)t] \\
&= \frac{1}{2}\cos[2\pi(\nu_1+\nu_2)t] + \frac{1}{2}\cos[2\pi(\nu_1-\nu_2)t] \\
&= \frac{1}{2}\left(\cos\left[2\pi\frac{t}{27.27}\right] + \cos\left[2\pi\frac{t}{300}\right]\right) \\
\nu_{sum} &= \frac{1}{27.27}\text{ Hz},\ \nu_{dif} = \frac{1}{300}\text{ Hz}
\end{aligned}$$

3.15 Introduction to Fourier Analysis

The motion resulting from the sum of two oscillations of different frequency is complex (*i.e.,* anharmonic) though still periodic since it repeats after a time defined by:

$$T_{\mathrm{mod}} = \frac{1}{\nu_{\mathrm{mod}}} = \frac{2\pi}{\omega_{\mathrm{mod}}} = \frac{4\pi}{\omega_1 - \omega_2}$$

As $\omega_1 \to \omega_2, T_{\mathrm{mod}}$ lengthens. In the limit, $T_{\mathrm{mod}} \to \infty$ and $T_{avg} \to T_1 = T_2$.

The addition of more oscillations of different frequencies produces more and more complex motion (less like harmonic motion). For example, consider this sum of harmonic oscillators:

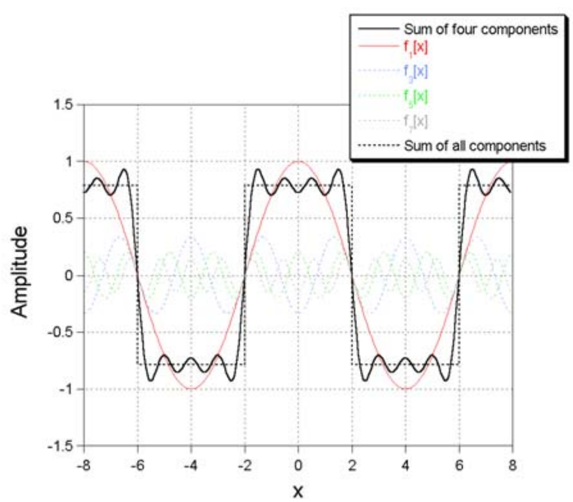
$$y\left[t\right] = \sum_{n=1,3,5,\cdots}^{\infty} \left(\pm\frac{1}{n}\cos\left[n\omega_0 t\right]\right)$$

For each succeeding term, the amplitude decreases and the frequency increases. The first term (*fundamental*) is:

$$\begin{aligned}
f_1\left[t\right] &= \cos\left[2\pi\frac{t}{8}\right] \\
f_2\left[t\right] &= 0 \\
f_3\left[t\right] &= -\frac{1}{3}\cos\left[2\pi\frac{t}{\left(\frac{8}{3}\right)}\right] \\
f_4\left[t\right] &= 0 \\
f_5\left[t\right] &= +\frac{1}{5}\cos\left[2\pi\frac{t}{\left(\frac{8}{5}\right)}\right]
\end{aligned}$$

Obviously, $y\left[t\right]$ is becoming less and less harmonic as more terms are added, and in fact is starting to look like a completely different function – a square wave. *Especially note that as higher frequency components are added (i.e.,* larger values of n*), the verticals become steeper and the edges become sharper.* Note also that the summation overshoots when transitioning from horizontal to vertical and vice versa. This is known as the Gibbs phenomenon, and the effect diminishes as more terms are added. As $N \to \infty$, the function $y\left[t\right]$ becomes a periodic square wave, which is quite dissimilar from the component functions.

This result illustrates the principle of *Fourier Analysis*, where we determine the set of sinusoidal constitutents that sum to create the function $f\left[t\right]$. The complementary operation of *Fourier synthesis* sums up a set of sinusoids to find the resultant.



Sum of sinusoids with specific different magnitudes and frequencies to produce a square wave.

(Virtually) every periodic function may be decomposed into a sum of sines and cosines with definite amplitudes, frequencies, and phases. The decomposition is unique, and is called the Fourier series representation, or the spectrum of the periodic function.

The spectrum is a representation of the amplitudes, frequencies, and phases of the sinusoidal components that superpose to create the function. Often, the term *spectrum* is used when *power spectrum* would be more accurate – it is the power (or energy, the squared magnitude of the component) that is plotted rather than the amplitude.

This concept should be quite familiar to you – the spectrum of white light is analogous. White light is a periodic function – it looks the same at all times. Spherical rain droplets act as prisms to disperse white light into its constituent components – the colors of the spectrum. The brightnesses of each color correspond to the energy of the component – brighter $\Longrightarrow$ more energy. The droplet prisms act as Fourier transformers since they derive the spectrum of the function As Newton showed, the spectrum can be transformed back to white light with another prism.